Journal of Computational Physics 229 (2010) 7401–7410

Contents lists available at ScienceDirect

Journal of Computational Physics

journal homepage: www.elsevier.com/locate/jcp

A new set of basis functions for the discrete geometric approach

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ARTICLE INFO

Article history:
Received 18 November 2009
Received in revised form 10 June 2010
Accepted 12 June 2010
Available online 19 June 2010

Keywords:
Discrete geometric approach (DGA)
Cell method
Finite integration technique (FIT)
Basis functions
Discrete constitutive equations
Discrete analogs of the hodge star operator
Non-orthogonal polyhedral dual grids

ABSTRACT

By exploiting the geometric structure behind Maxwell's equations, the so called discrete geometric approach allows to translate the physical laws of electromagnetism into discrete relations, involving circulations and fluxes associated with the geometric elements of a pair of interlocked grids: the primal grid and the dual grid.

To form a finite dimensional system of equations, discrete counterparts of the constitutive relations must be introduced in addition. They are referred to as constitutive matrices which must comply with precise properties (symmetry, positive definiteness, consistency) in order to guarantee the stability and consistency of the overall finite dimensional system of equations.

The aim of this work is to introduce a general and efficient set of vector functions associated with the edges and faces of a *polyhedral* primal grids or of a dual grid obtained from the barycentric subdivision of the boundary of the primal grid; these vector functions comply with precise specifications which allow to construct stable and consistent discrete constitutive equations for the discrete geometric approach in the framework of an energetic method.

1. Introduction

Maxwell equations formulated in terms of partial differential equations are commonly and conveniently discretized by means of finite elements techniques which produce algebraic relations through a sound mathematical machinery based, for example, on the Galerkin's method [2].

However, in the recent years, a less conventional method has gained interest within the computational electromagnetic community, developed by Yee [1] with a FDTD method, by Clemens and Weiland [3] with the Finite Integration Technique (FIT), by Tonti [5] with the cell method (CM), by Bossavit [6] with a reinterpretation of finite element method (FEM) and by present authors [7] with the Discrete Geometric Approach (DGA).

In the DGA approach emphasis is put on the geometric structure behind Maxwell's equations. The physical laws of electromagnetism are recognized to be balance equations and they are *exactly* translated into algebraic relations involving circulations and fluxes (of the electromagnetic field quantities) associated with geometric elements (nodes, edges, faces and volumes) of a pair of interlocked grids (primal grid-dual grid). Discrete counterparts of the constitutive relations between field quantities are also introduced; they are *approximate* algebraic operators (matrices) which map circulations along edges of the primal grid onto fluxes through faces of the dual grid or viceversa, and involve the material properties and the metric notions related to the geometry of the grids.

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0021-9991/$ - see front matter
doi:10.1016/j.jcp.2010.06.023

As a known result [8,9] in order to ensure the consistency and the stability of the overall final system of algebraic equations, the discrete constitutive relations are required to satisfy stability and consistency properties. The stability requirement prescribes that the constitutive matrices are *symmetric* and *positive definite*. The consistency requirement prescribes that constitutive matrices *exactly* map circulations onto fluxes or viceversa, at least for element-wise *uniform* fields.

Stable and consistent discrete constitutive equations can be naturally constructed for pairs of orthogonal Cartesian grids as shown by Yee [1] and Clemens and Weiland [3]. Besides, as shown by Bossavit [11], the mass matrices constructed by means of edge and face elements introduced by Whitney and generalized by Nedelec [18,19], satisfy both the stability and consistency properties required by DGA [11], for pairs of grids in which the primal is composed of tetrahedra and the dual grid is obtained according to the barycentric subdivision of the primal. This result unfortunately does not hold in general for edge and face elements relative to different geometries. For instance, present authors have proven in [14] that Whitney's elements for generic hexahedral primal grids do *not* satisfy the consistency property required by DGA, for any choice of the dual grid.

By the introduction of a novel set of edge and face vector functions combined with an energetic approach [12], the present authors were able in [7] to derive novel constitutive matrices satisfying both the consistency and stability properties required by DGA, not only for tetrahedra but also for (oblique) prisms with triangular base.

However, for pairs of grids where the primal grid is based on general polyhedra, useful in many applications, no constitutive matrices satisfying the consistency and stability properties required by DGA were reported in literature, as far as the authors know. It is here noted that some approaches can be found in literature for generating discrete counterparts of constitutive relations over polyhedral grids, such the mimetic finite differences [20–22] or the mixed finite elements [23]. However all these methods do not lead in general to discrete constitutive relations satisfying the consistency property required by DGA. The present authors did first attempts to fill in this gap with papers [13,14].

The novelty content of this work is the introduction of four new general sets of vector functions for *polyhedral* primal grids associated with edges and faces of both the primal and of the dual grids. They are constructed directly in terms of the geometric elements (edges and faces) of the primal and of the dual grids. These vector functions are designed in such a way to comply with the requirements of the *energetic approach* introduced by the authors [12] for deriving discrete constitutive equations which ensure the consistency and stability properties required by DGA.

The functions here proposed belong to a class which, as recently shown by some of the present authors [24], theoretically ensures the convergence of the solution of discrete equations to the exact solution of the continuous problem. It is here noted that this class of functions, unlike Whitney's and Nedelec's basis functions, do not satisfy any curl-conforming or div-conforming properties. Thus in DGA, by using a pair of dual grids instead of a single grid, convergence can be guaranteed by using basis functions which do not satisfy all the regularity conditions of Whitney's and Nedelec's basis functions.

Numerical experiments will demonstrate that the novel discrete constitutive matrices can be computed easily and in a very efficient way leading to accurate approximations of the solution of a magnetostatic problem proposed as an example.

2. Pair of interlocked grids and geometric properties

Without losing generality, we will focus on a primal grid consisting of a single polyhedron v, Fig. 1.

The geometric elements of the primal grid are nodes, edges, faces and the volume v. We denote a primal edge with e_i, where $i = 1,\dots,L$, L being the number of edges of v and a primal face with f_j, where $j = 1,\dots,F$, F being the number of faces

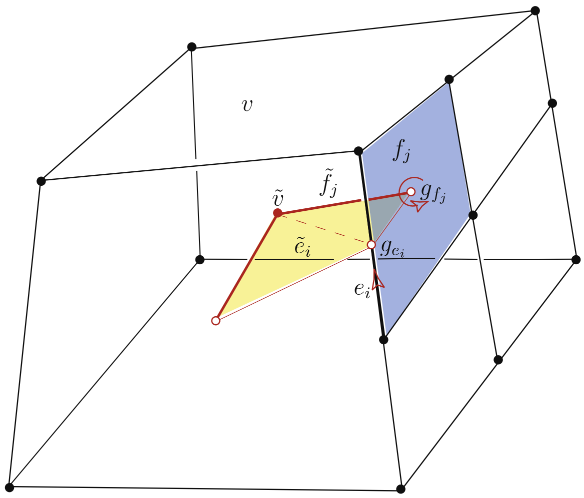

Fig. 1. A polyhedron v is evidenced, together with a primal edge e_i and its dual geometric entity the dual face $\tilde{e}_i$, a primal face f_j and its dual geometric entity the dual edge $\tilde{f}_j$. Moreover the barycenters g_{e_i} and g_{f_j} of edge e_i and of face f_j are shown respectively.

of v. The geometric entities of the primal grid like e_i, f_j are provided with an inner orientation [4,10]; For example in Fig. 1 the arrows indicate a possible choice of inner orientations for edge e_i and face f_j respectively.

Interlocked with the primal grid, a dual grid is introduced; each geometric entity of the dual grid is in a one-to-one correspondence (duality pairing) with a geometric element of the primal grid. The dual of a primal volume v is the dual node denoted as $\tilde{v}$, where symbol "$\sim$" acts on the geometric entity yielding its corresponding dual[1]; similarly the dual of a primal edge e_i is the dual face $\tilde{e}_i$ and the dual of a primal face f_j is the dual edge $\tilde{f}_j$, Fig. 1. We assume *arbitrary* the location of the dual node $\tilde{v}$ within v, while the geometric construction of dual edges and dual faces is based on the *barycentric subdivision* of the *boundary* of v as follows. We introduce the barycenters $g_{e_i}, \ g_{f_j}$ of e_i, f_j respectively; a dual edge $\tilde{f}_j$ is a segment whose boundary are the nodes $\tilde{v}, \ g_{f_j}$. A dual face $\tilde{e}_i$ is a quadrilateral surface made of the union of a pair of triangles; one of the triangles has the nodes $\tilde{v}, g_{e_i}, \ g_{f_j}$ as vertices, similarly for the other, Fig. 1. The only geometric hypothesis on v we assume is that all edges $\tilde{e}_i$ and faces $\tilde{f}_j$ are contained in v. It is straightforward to verify as in [13] that a sufficient, but not a necessary, condition for that is that v is convex.

Dual edge $\tilde{f}_j$ and dual face $\tilde{e}_i$ are endowed with outer orientation [4,8], in such a way that each of the pairs $(e_i, \tilde{e}_i)$, $(f_j, \tilde{f}_j)$ is oriented in a congruent way.

Vector e_i, denoted in roman type, is the edge vector[2] associated with the edge e_i. Vector f_j is the face vector associated with the face f_j defined as $\mathrm{f}_i = \int_{f_j} \mathrm{n} ds$, where n is the unit vector normal to and oriented as f_j. Similarly vector $\tilde{\mathrm{e}}_i$ is the face vector associated with dual face $\tilde{e}_i$ and $\tilde{\mathrm{f}}_j$ is the edge vector associated with dual edge $\tilde{f}_j$.

2.1. Fundamental geometric properties

To simplify the notation hereafter we introduce symbol r_i to denote one of the following geometric entities $\{e_i, f_j, \tilde{e}_i, \tilde{f}_j\}$ of the primal or of the dual grid and symbol r_i to denote the corresponding vector in the set $\{\mathrm{e}_i, \mathrm{f}_j, \tilde{\mathrm{e}}_i, \tilde{\mathrm{f}}_j\}$. We denote with $\tilde{r}_i$ the geometric entity dual of r_i, and with $\tilde{\mathrm{r}}_i$ its corresponding vector in the set $\{\mathrm{e}_i, \ \mathrm{f}_j, \tilde{\mathrm{e}}_i, \tilde{\mathrm{f}}_j\}$.

Thence, we have $\tilde{\mathrm{r}}_i \cdot \mathrm{r}_i > 0$, with $i = 1, \ldots, R$ and R is any in $\{L, F\}$.

Hereafter, we will denote with

$$\mathrm{T}_i = \tilde{\mathrm{r}}_i \otimes \mathrm{r}_i, \tag{1}$$

the double tensor T_i obtained from the tensor product $\otimes$ of the two vectors r_i and its dual $\tilde{\mathrm{r}}_i$, with $i = 1, \ldots, R$; its Cartesian components are $(\mathrm{T}_i)_{hk} = (\tilde{\mathrm{r}}_i \otimes \mathrm{r}_i)_{hk} = (\tilde{\mathrm{r}}_i)_h (\mathrm{r}_i)_k$, where $(\mathrm{r}_i)_h$ is the hth Cartesian component of r_i, with $h, k = 1, \ldots, 3$. The trace of T_i is

$$t_i = \mathrm{tr}(\mathrm{T}_i) = \tilde{\mathrm{r}}_i \cdot \mathrm{r}_i, \tag{2}$$

where "$\cdot$" is the usual inner product between vectors. The product $\mathrm{T}_i\mathrm{x}$ between double tensor T_i and a generic vector x is a vector and we write

$$\mathrm{T}_i \mathrm{x} = (\mathrm{r}_i \cdot \mathrm{x}) \tilde{\mathrm{r}}_i. \tag{3}$$

The identity tensor is denoted with I and Ix = x holds.

Provided that the dual grid is constructed according to the barycentric subdivision of the *boundary* of v, in papers [13,15], we proved the following geometric identities

$$\sum_{i=1}^{L} \mathrm{e}_i \otimes \tilde{\mathrm{e}}_i = |v| \mathrm{I}, \quad \sum_{i=1}^{F} \mathrm{f}_j \otimes \tilde{\mathrm{f}}_j = |v| \mathrm{I}, \tag{4}$$

where $|v|$ is the volume of v. Identities (4) can now be conveniently rewritten as

$$\mathrm{T} = \sum_{i=1}^{R} \mathrm{T}_i = \sum_{i=1}^{R} \tilde{\mathrm{r}}_i \otimes \mathrm{r}_i = |v| \mathrm{I}. \tag{5}$$

Obviously, tensor T is symmetric and tr (T) = $3|v|$ holds.

2.2. Partition of the polyhedron

We introduce a partition of polyhedron v into a number of subregions τ_i^r in a one-to-one correspondence with the geometric element r_i, with $i = 1, \ldots, R$. Precisely, subregion $\tau_i^e = \tau_i^{\tilde{e}}$ is shown in Fig. 2(a), with $i = 1, \ldots, L$; it is a polyhedral region individuated by e_i, or equivalently $\tilde{e}_i$, formed by a pair of tetrahedra, each of which having as vertices the dual node $\tilde{v}$, the pair of nodes bounding e_i, and one of the barycenters of the two primal faces having e_i in common. Fig. 2(b), shows subregion $\tau_j^f = \tau_j^{\tilde{f}}$, with $j = 1, \ldots, F$; it is a polyhedral region individuated by f_j or equivalently $\tilde{f}_j$, formed by a pyramid having as base the f_j face and as apex the dual node $\tilde{v}$.

[1] The dual of the dual yields the geometric entity itself.

[2] Its amplitude, direction and orientation coincide with the length, direction and orientation of e_i respectively.

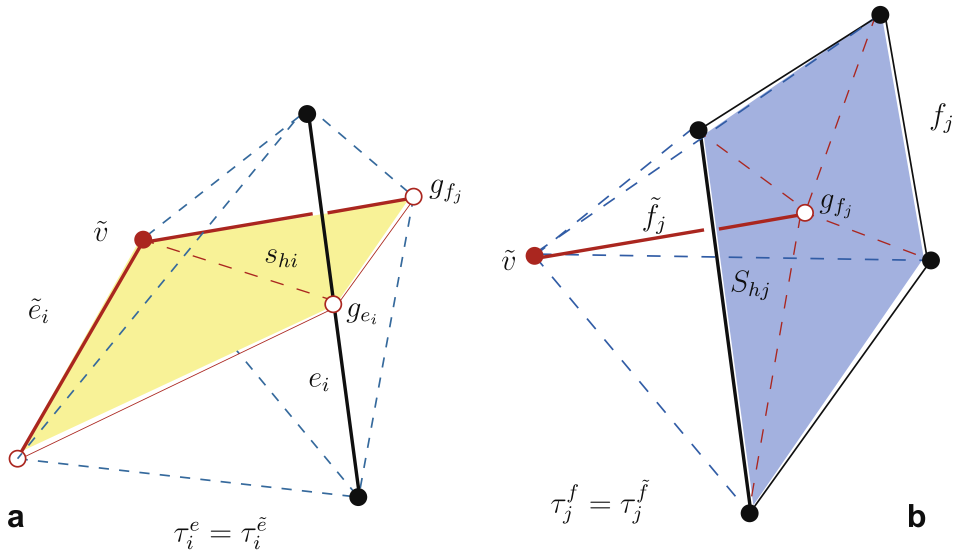

Fig. 2. Subregions $\tau_i^e = \tau_i^{\tilde{e}}$ and $\tau_j^f = \tau_j^{\tilde{f}}$ are shown in detail.

Lemma 1. *We have that*

$$t_i = \mathrm{tr}(\mathrm{T}_i) = 3|\tau_i^r| \tag{6}$$

holds, where $|\tau_i^r|$ *is the volume of the subregion* τ_i^r.

Proof. Firstly, let us consider the case of a subregion $\tau_i^e = \tau_i^{\tilde{e}}$, Fig. 2(a). For dual face $\tilde{e}_i$, we write $\tilde{e}_i = \cup_{h=1}^{2} s_{hi}$, where s_{hi} is one of the two triangular portions forming $\tilde{e}_i$; In terms of area vectors we write $\tilde{\mathrm{e}}_i = \sum_{h=1}^{2} \mathrm{s}_{hi}$. Then, from the formula giving the volume of a tetrahedron,

$$\mathrm{e}_i \cdot \tilde{\mathrm{e}}_i = \sum_{h=1}^{2} \mathrm{s}_{hi} \cdot \ \mathrm{e}_i = \sum_{h=1}^{2} 3|V_h| = 3|\tau_i^e| \tag{7}$$

holds, where $|V_h|$ is the volume of one of the two tetrahedra V_h forming τ_i^e, Fig. 3. Then from (2) the thesis follows.

Secondly let us consider the case of a subregion $\tau_j^f = \tau_j^{\tilde{f}}$, Fig. 2(b). From the formula expressing the volume of a pyramid

$$\mathrm{f}_j \cdot \tilde{\mathrm{f}}_j = 3|\tau_j^f| \tag{8}$$

holds. Then from (2) the thesis follows. □

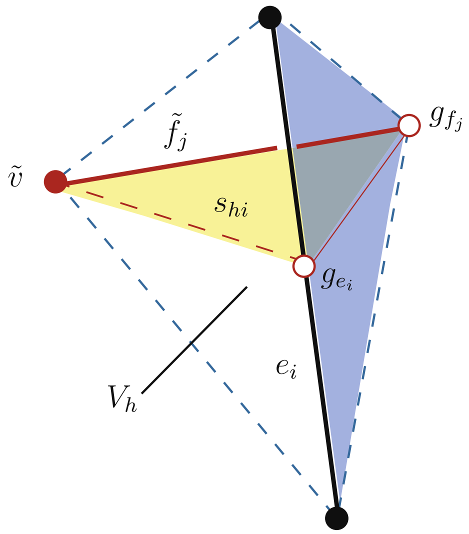

Fig. 3. Detail of a tetrahedron V_h forming the subregions and $\tau_j^e = \tau_j^{\tilde{e}}$.

3. Construction of the basis vector functions

We consider a vector field $\mathrm{x}(p)$ in v; For example, an electric field E or a current density vector J within the polyhedron. The integral quantity

$$X_i^r = \int_{r_i} \mathrm{x}(p) \cdot d\mathrm{r} \tag{9}$$

represents either a circulation or a flux of the field $\mathrm{x}(p)$ provided that the geometric entity r_i is an edge or a face respectively, X_i^r is often referred to as Degree of Freedom; Symbol "$d\mathrm{r}$" stands for $d\mathrm{l}$ or $d\mathrm{s}$ according to a line or surface integration is performed respectively. For example, (9) yields the usual electro-motive force (e.m.f.) $U_i = \int_{e_i} \mathrm{E}(p) \cdot d\mathrm{l}$ along a primal edge e_i or the current $I_i = \int_{\tilde{e}_i} \mathrm{J}(p) \cdot d\mathrm{s}$ crossing a dual face $\tilde{e}_i$.

Now let us assume that $\mathrm{x}(p)$ is uniform in v and thus x does not depend on p. Then by right multiplying (5) by x, from (3), (5) and (9), since $X_i^r = \mathrm{x} \cdot \mathrm{r}_i$, we obtain

$$\mathrm{x} = \frac{1}{|v|} \sum_{i=1}^{R} X_i^r \tilde{\mathrm{r}}_i. \tag{10}$$

Besides, multiplying on the right by x both members of the identity

$$\mathrm{I} = \frac{\mathrm{T}_j}{t_j} + \left(\mathrm{I} - \frac{\mathrm{T}_j}{t_j} \right) \tag{11}$$

and using (3), we have that

$$\mathrm{x} = \frac{X_j^r}{t_j} \tilde{\mathrm{r}}_j + \left(\mathrm{I} - \frac{\mathrm{T}_j}{t_j} \right) \mathrm{x}, \tag{12}$$

holds in subregion τ_j^r, with $j = 1, \ldots, R$. Thus by substituting (10) for x in the right hand side of (12), we obtain

$$\mathrm{x} = \sum_{i=1}^{R} \left(\frac{\tilde{\mathrm{r}}_j}{t_j} \delta_{ij} + \left(\mathrm{I} - \frac{\mathrm{T}_j}{t_j} \right) \frac{\tilde{\mathrm{r}}_i}{|v|} \right) X_i^r, \tag{13}$$

where δ_{ij} is the Kronecker symbol, or equivalently

$$\mathrm{x} = \sum_{i=1}^{R} \mathrm{v}_i^r(p) X_i^r, \tag{14}$$

in which

$$\mathrm{v}_i^r(p) = \frac{\tilde{\mathrm{r}}_j}{t_j} \delta_{ij} + \left(\mathrm{I} - \frac{\mathrm{T}_j}{t_j} \right) \frac{\tilde{\mathrm{r}}_i}{|v|}, \quad \text{for each } p \in \tau_j^r, \quad \text{with } j = 1, \ldots, R. \tag{15}$$

Quantities $\mathrm{v}_i^r(p)$, with $i = 1, \ldots, R$, derived in this way, are vector functions, piece-wise uniform in v and uniform in each subregion τ_j^r with $j = 1, \ldots, R$. Moreover they satisfy the following three properties, fundamental to construct constitutive matrices for the DGA, as outlined by the authors in [12].

Property 1. *The functions $v_i^r(p)$, with $i = 1, \ldots, R$ are linearly independent, and are such that*

$$\int_{r_j} \mathrm{v}_i^r(p) \cdot d\mathrm{r} = \delta_{ij}, \tag{16}$$

holds, for $i, j = 1, \ldots, R$.

Proof. In the subregion τ_j^r adjacent to r_j, we write

$$\int_{r_j} \mathrm{v}_i^r(p) \cdot d\mathrm{r} = \left(\frac{\tilde{\mathrm{r}}_j}{t_j} \delta_{ij} + \left(\mathrm{I} - \frac{\mathrm{T}_j}{t_j} \right) \frac{\tilde{\mathrm{r}}_i}{|v|} \right) \cdot \mathrm{r}_j = \delta_{ij} + \left(\frac{\tilde{\mathrm{r}}_i}{|v|} - \frac{\mathrm{r}_j \cdot \tilde{\mathrm{r}}_i}{t_j |v|} \tilde{\mathrm{r}}_j \right) \cdot \mathrm{r}_j = \delta_{ij} + \left(\frac{\mathrm{r}_j \cdot \tilde{\mathrm{r}}_i}{|v|} - \frac{\mathrm{r}_j \cdot \tilde{\mathrm{r}}_i}{|v|} \right) = \delta_{ij}.$$

In the second equality (3) has been applied, while in the first and last equalities we used (2). Thus (16) holds. As a consequence functions $\mathrm{v}_i^r(p)$, with $i = 1, \ldots, R$ are linearly independent. □

Property 2. *The functions $v_i^r(p)$, with $i = 1, \ldots, R$ allow to represent exactly a uniform vector field from its Degrees of Freedom, according to (14).*

Proof. The thesis straightforwardly follows from (13).

Property 3. *The consistency condition*

$$\int_v \mathrm{v}_i^r(p)dv = \tilde{\mathrm{r}}_i, \tag{17}$$

holds, with i = 1,...,R.

Proof. We compute

$$\int_v \mathrm{v}_i^r(p)dv = \sum_{j=1}^{R}\int_{\tau_j^r}\mathrm{v}_i^r(p)dv = \sum_{j=1}^{R}\left(\frac{\tilde{\mathrm{r}}_j}{t_j}\delta_{ij} + \left(\mathrm{I} - \frac{\mathrm{T}_j}{t_j}\right)\frac{\tilde{\mathrm{r}}_i}{|v|}\right)|\tau_j^r| = \sum_{j=1}^{R}\tilde{\mathrm{r}}_j\frac{|\tau_j^r|}{t_j}\delta_{ij} + \left(\sum_{j=1}^{R}|\tau_j^r|\right)\frac{\tilde{\mathrm{r}}_i}{|v|} - \frac{1}{3}\left(\sum_{j=1}^{R}\mathrm{T}_j\right)\frac{\tilde{\mathrm{r}}_i}{|v|}$$
$$= \frac{1}{3}\tilde{\mathrm{r}}_i + \tilde{\mathrm{r}}_i - \frac{1}{3}\tilde{\mathrm{r}}_i = \tilde{\mathrm{r}}_i,$$

where, in the second equality we used Lemma 1 and in the third equality we used the identity (5). □

4. Constitutive matrix

We consider a single polyhedron v, where a pair of vector fields x, y exists, related by a constitutive relation

$$y = mx, \tag{18}$$

m being a double tensor, representing the material property, assumed to be symmetric positive definite and homogeneous in v.

Now, we focus on the pairs of geometric elements r_i, $\tilde{r}_i$, one dual of the other, with $i = 1,\ldots,R$ and we introduce the corresponding pair of Degrees of freedom $X_i^r = \int_{r_i} x \cdot d\mathrm{r}$, $Y_i^{\tilde{r}} = \int_{\tilde{r}_i} y \cdot d\mathrm{r}$. We denote in boldface type the arrays $\mathbf{X}^r$, $\mathbf{Y}^{\tilde{r}}$, of dimension R, formed by X_i^r, $Y_i^{\tilde{r}}$ respectively, and we introduce the discrete counterpart of (18) in v as

$$\mathbf{Y}^{\tilde{r}} \cong \mathbf{M}^{r\tilde{r}}(m)\mathbf{X}^r, \tag{19}$$

where $\mathbf{M}^{r\tilde{r}}(m)$ is a *constitutive matrix* of dimension R mapping the Degrees of Freedom array $\mathbf{X}^r$ onto $\mathbf{Y}^{\tilde{r}}$ associated with geometric elements r_i, $\tilde{r}_i$, one dual of the other respectively; (19) holds only approximately, yielding the well known *constitutive error* affecting the overall discrete problem [8].

As shown in [8,9,11] the aim is to construct a constitutive matrix $\mathbf{M}^{r\tilde{r}}(m)$ which complies with the following requirements: (i) it is symmetric, (ii) it is positive definite and (iii) it is such that (19) holds *exactly* at least for a pair of uniform fields x, y in v. It is well known that the requirements (i) and (ii) are fundamental to guarantee the stability of the discretized equations while the last requirement (iii) guarantees the consistency of the discretized equations in the DGA.

In order to comply with these requirements, we will resort to the so called *energetic approach* proposed in [7,12,14,15] which relies solely on Properties 1, 2 and 3 for the vector basis function $\mathrm{v}_i^r(p)$ with $i = 1,\ldots,R$. According to such energetic approach, the number

$$M_{ij}^{r\tilde{r}}(m) = \int_v \mathrm{v}_i^r(p)\cdot m\ \mathrm{v}_j^r(p)dv \quad i,j = 1,\ldots,R \tag{20}$$

is the i, j entry of the constitutive matrix $\mathbf{M}^{r\tilde{r}}(m)$ complying with the requirements (i) (ii) and (iii).

Interestingly, the integration in (20) can be performed exactly, the vector basis function $\mathrm{v}_i^r(p)$ being piece-wise uniform. Given points $p_k \in \tau_k^r$, with $k = 1,\ldots,R$, we obtain

$$M_{ij}^{r\tilde{r}}(m) = \sum_{k=1}^{R}\mathrm{v}_i^r(p_k)\cdot m\mathrm{v}_j^r(p_k)\frac{t_k}{3}, \tag{21}$$

where we used Lemma 1.

We also note that the proposed approach suggests two alternative ways to compute a constitutive matrix mapping $\mathbf{X}^r$ onto $\mathbf{Y}^{\tilde{r}}$, satisfying Properties 1, 2 and 3. One way is provided by matrix $\mathbf{M}^{r\tilde{r}}(m)$ whose entries are $M_{ij}^{r\tilde{r}}(m)$ as given by (20). Let us now substitute in (20) $\tilde{r}$ for r and m^{-1} for m, obtaining

$$M_{ij}^{\tilde{r}r}(m^{-1}) = \int_v \mathrm{v}_i^{\tilde{r}}(p)\cdot m^{-1}\mathrm{v}_j^{\tilde{r}}(p)dv \quad i,j = 1,\ldots,R, \tag{22}$$

the entries of a matrix $\mathbf{M}_{ij}^{\tilde{r}r}(m^{-1})$ mapping $\mathbf{Y}^r$ onto $\mathbf{X}^{\tilde{r}}$; Then $\left(\mathbf{M}_{ij}^{\tilde{r}r}(m^{-1})\right)^{-1}$ is a new matrix mapping $\mathbf{X}^r$ onto $\mathbf{Y}^{\tilde{r}}$. As it can be verified by examples, matrices $\mathbf{M}^{r\tilde{r}}(m)$ and $\left(\mathbf{M}_{ij}^{\tilde{r}r}(m^{-1})\right)^{-1}$ are in general different matrices and thus provide different counterparts of constitutive relations.

5. Numerical results

The proposed constitutive matrices can be conveniently used to solve various typologies of problems arising in computational physics. In order to test the proposed constitutive matrices, we focus on reference magnetostatic problems which are solved by using a pair of complementary geometric formulations. The two formulations are based on circulation on primal edges of a magnetic vector potential A and on a magnetic scalar potential Ω defined in primal nodes respectively; see for example [16] for a detailed description. The magnetostatic formulations need the reluctance and the permeance constitutive matrices, which are described in the following subsections.

5.1. Reluctance matrix using v_i^f

The reluctance matrix $\boldsymbol{\nu} = \mathbf{M}^{f\tilde{f}}(\nu)$ for polyhedron v relates the magnetic induction flux $\Phi_i = X_i$ associated with f_i with the magneto-motive forces (m.m.f.s) $F_i = Y_i$ associated with $\tilde{f}_i$, with $i = 1,\ldots,F$, where ν is the uniform reluctivity in v; $\dim(\boldsymbol{\nu}) = F$ holds. The entries of $\boldsymbol{\nu}$ are

$$\nu_{ij} = M_{ij}^{f\tilde{f}}(\nu) = \int_v \mathrm{v}_i^f \cdot \nu \mathrm{v}_j^f \, dv.$$

5.2. Reluctance matrix using $\mathrm{v}_i^{\tilde{f}}$

As a first step we construct the permeance matrix $\tilde{\boldsymbol{\mu}} = \mathbf{M}^{\tilde{f}f}(\nu^{-1})$ for the polyhedron v relating the m.m.f.s $F_i = X_i$ with the magnetic induction fluxes $\Phi_i = Y_i$, with $i = 1,\ldots,F$; $\dim(\tilde{\boldsymbol{\mu}}) = F$ holds. The entries of $\tilde{\boldsymbol{\mu}}$ are

$$\tilde{\mu}_{ij} = M_{ij}^{\tilde{f}f}(\nu^{-1}) = \int_v \mathrm{v}_i^{\tilde{f}} \cdot \mu \mathrm{v}_j^{\tilde{f}} \, dv,$$

where $\mu = \nu^{-1}$ is the uniform permeability in v. As a second step, we invert $\tilde{\boldsymbol{\mu}}$ obtaining the reluctance matrix $\tilde{\boldsymbol{\nu}} = \tilde{\boldsymbol{\mu}}^{-1} = (\mathbf{M}^{\tilde{f}f}(\nu^{-1}))^{-1}$; $\tilde{\boldsymbol{\nu}}$ is an alternative constitutive matrix with respect to $\boldsymbol{\nu}$.

5.3. Permeance matrix using v_i^e

The permeance matrix $\boldsymbol{\mu} = \mathbf{M}^{e\tilde{e}}(\mu)$ for polyhedron v relates the m.m.f.s $F_i = X_i$ associated with e_i with the magnetic induction fluxes $\Phi_i = Y_i$ associated with $\tilde{e}_i$, with $i = 1,\ldots,L$; $\dim(\boldsymbol{\mu}) = L$ holds. The entries of $\boldsymbol{\mu}$ are

$$\mu_{ij} = M_{ij}^{e\tilde{e}}(\mu) = \int_v \mathrm{v}_i^e \cdot \mu \mathrm{v}_j^e \, dv.$$

5.4. Permeance matrix using $\mathrm{v}_i^{\tilde{e}}$

As a first step, we construct matrix $\tilde{\boldsymbol{\nu}} = \mathbf{M}^{\tilde{e}e}(\mu^{-1})$ for polyhedron v relating $\Phi_i = X_i$ with $F_i = Y_i$; $\dim(\tilde{\boldsymbol{\nu}}) = L$ holds. The entries of $\tilde{\boldsymbol{\nu}}$ are

$$\tilde{\nu}_{ij} = M_{ij}^{\tilde{e}e}(\mu^{-1}) = \int_v \mathrm{v}_i^{\tilde{e}} \cdot \nu \mathrm{v}_j^{\tilde{e}} \, dv.$$

As a second step, we invert $\tilde{\boldsymbol{\nu}}$ obtaining the permeance matrix $\tilde{\boldsymbol{\mu}} = (\mathbf{M}^{\tilde{e}e}(\mu^{-1}))^{-1} = \tilde{\boldsymbol{\nu}}^{-1}$. Again, $\tilde{\boldsymbol{\mu}}$ is an alternative constitutive matrix with respect to $\boldsymbol{\mu}$.

5.5. Consistency test

To test the consistency numerically, firstly we consider a static case, where the actual fields are uniform.

In the domain D — a cube of unitary edge — we constructed an undistorted primal complex consisting of $3 \times 3 \times 3$ cubical elements, see Fig. 4(a).

By displacing some nodes, we obtain a new deformed primal complex $\mathscr{K}$ made of 27 hexahedra, see Fig. 4(b). Then, we apply the *subgridding technique* [17] by subdividing the central hexahedron in 64 hexahedra and the hexahedron below in 8 hexahedra, see Fig. 4(c). Thus, the final grid in formed by 97 cells, 369 faces, 478 edges, and 195 nodes.

The boundary conditions have been set in order to generate in D a uniform magnetic induction field B of amplitude 1T and pointing down the vertical axis.

The linear systems of equations arising from the magnetostatic problem discretized by the DGA using both complementary formulations [16] on the polyhedral primal complex $\mathscr{K}$ are solved using the reluctance and permeance constitutive matrices described in Sections 5.1, 5.2 and 5.3, 5.4, respectively.

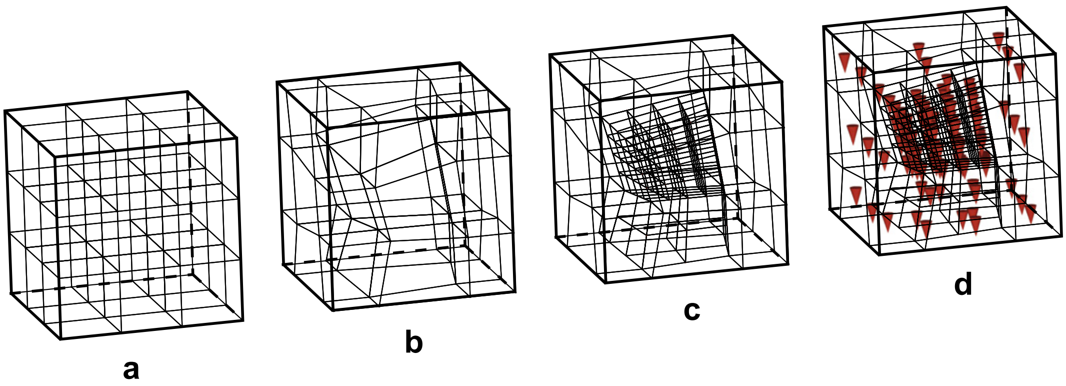

Fig. 4. (a) The unitary cube D is partitioned into 27 cubes. (b) Some of the nodes are displaced, obtaining 27 hexahedra. (c) The subgridding is applied, subdividing two hexahedra in 64 and 8 hexahedra, respectively. (d) The uniform field, solution of the magnetostatic problem, is interpolated exactly.

It is possible to see that the uniform field solution of the magnetostatic problem is interpolated exactly in the whole domain D, see Fig. 4(d), using both formulations and constitutive matrices constructed by means of basis functions of the primal or dual complex.

5.6. A magnetostatic problem

We will now move to a non-uniform field problem.

The reluctance and permeance constitutive matrices described in Sections 5.1, 5.2 and 5.3, 5.4, respectively, are used to solve a reference magnetostatic problem consisting in a sphere of radius $R = 0.35$ m made of linear magnetic medium with relative permeability $\mu_r = 1000$ immersed in air. Only 1/8 of the problem is meshed with a grid made of 26,121 polyhedra, 85,507 faces, 88,317 edges, and 28,932 nodes.

The primal grid is obtained by means of the subgridding of an initial coarse hexahedral grid and by cutting each hexahedra intersecting the spherical surface by means of triangles [17], as shown in Fig. 5.

In Fig. 6, the polyhedral grid of the sphere and the triangular faces bounding the spherical surface are represented. This kind of polyhedral elements provide a very good tessellation of the spherical surface avoiding the staircase effect.

An external uniform induction field $\mathrm{B} = B_z\hat{z}$, $B_z = 1$ T being the field component along the vertical axis, is enforced by boundary conditions.

The magnetostatic problem using the A formulation [16] consists of 88,317 unknowns and it is solved in about 3.1 s and 5.3 s, using the $\boldsymbol{\nu}$ and $\tilde{\boldsymbol{\nu}}$, respectively.

The magnetostatic problem using the Ω formulation [16] consists of 28,932 unknowns and it is solved in about 0.9 s and 1.8 s, using the $\boldsymbol{\mu}$ and $\tilde{\boldsymbol{\mu}}$, respectively.

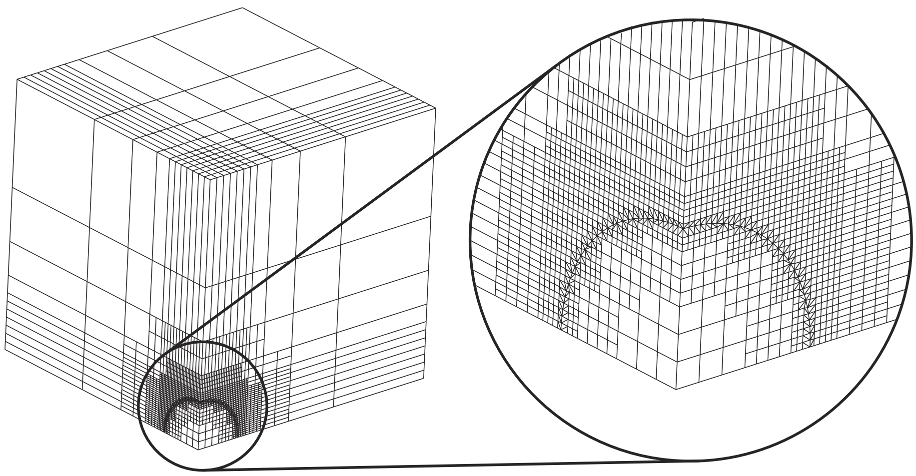

Fig. 5. The trace on the boundary of the domain box of the grid obtained by the subgridding of an initial hexahedral grid and successive hexahedral splitting is shown.

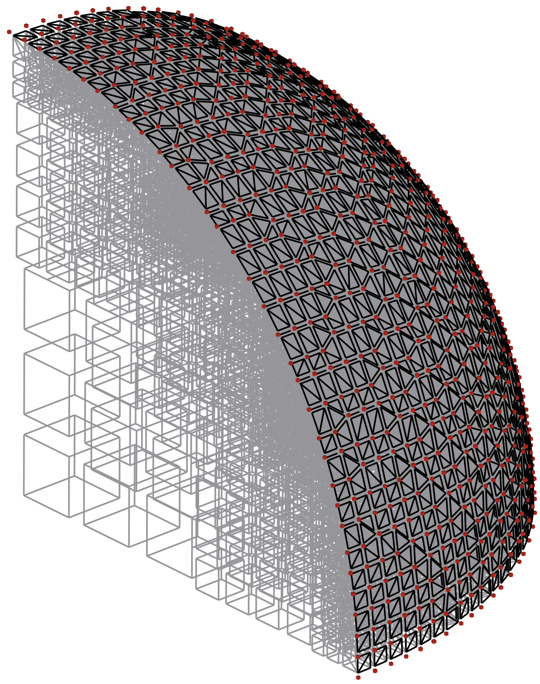

Fig. 6. Detail of the polyhedral grid and its trace on the interface surface between sphere and air.

Fig. 7 shows the computed component B_z along the z axis of B using both complementary magnetostatic formulations, along a number of sample points on a horizontal line. In the same Figure, B_z has been compared with the analytical solution showing very good agreement.

Table 1 contains the numbers of iterations needed by the Conjugate Orthogonal Conjugate Gradient (COCG) method to converge at the solution of the linear systems of equations together with the error in energy norm, evaluated by using

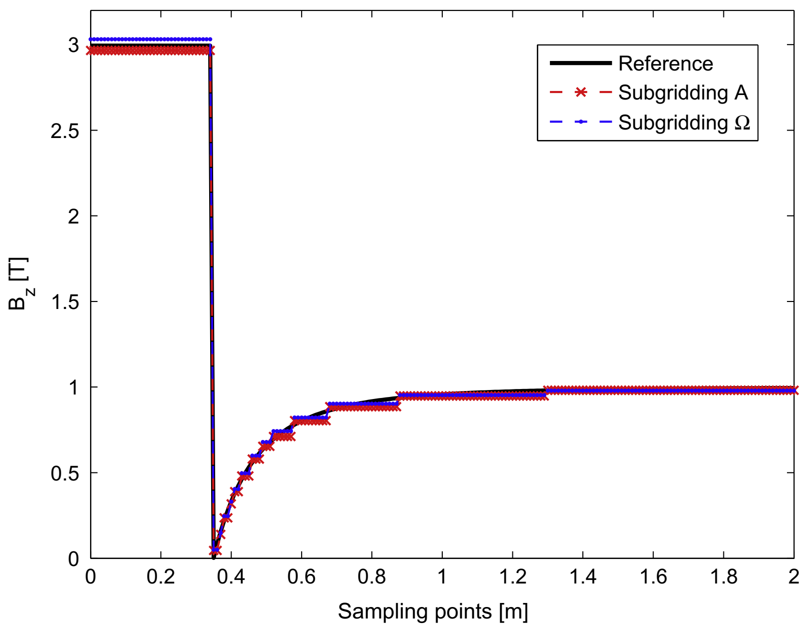

Fig. 7. The computed B_z components by the complementary magnetostatic formulations on a number of sample points along a line are shown and compared with the analytical solutions.

Table 1
COCG iterations and errors in energy norm for each proposed constitutive matrix.

Constitutive matrix	ν	$\tilde{\nu}$	μ	$\tilde{\mu}$
Formulation [16]	A	A	Ω	Ω
Unknowns	88,317	88,317	28,932	28,932
COCG iterations	106	193	74	157
ϵ_B [%]	0.60	0.86	0.43	0.78

$$\epsilon_B = \sqrt{\frac{\int_D \nu |\mathrm{B} - \mathrm{B}_{ref}|^2 dV}{\int_D \nu |\mathrm{B}_{ref}|^2 dV}},$$

which also shows a very good agreement with the analytical solution.

6. Conclusion

New vector basis functions, which allow to construct stable and consistent discrete constitutive equations for the Discrete Geometric Approach, have been introduced. The computation of the resulting constitutive matrices is computationally efficient, being based on a closed-form geometric construction. The results obtained by such constitutive matrices, considering a magnetostatic benchmark problem, are in good agreement with the analytical solution.

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